Let $h^2(R^2c^2-l^2r^2)=b^2l^4$, $R^2c/l^2=d$.

Then $A=(l/h)\sqrt{\{(r^2-u^2)[b^2+(u+d)^2]\}}$.

This can be integrated by a process similar to that employed by Professor Finkel in Vol. V, No. 1, pages 20, 21.

Let $Rc-lr$, then $A=\frac{1}{h}(lu+Rr)\sqrt{(r^2-u^2)}=\frac{R}{rh}(cu+r^2)\sqrt{(r^2-u^2)}$.

$$\therefore\ V=\frac{2R}{hr}\int_{-r}^{r}(cu+r^2)\sqrt{(r^2-u^2)}du+\frac{2\,Rr}{3h}\int_{-r}^{r}\frac{(u+c)^2du}{\sqrt{(r^2-u^2)}}$$

$$=\frac{2\pi\,Rr}{3h}(2r^2+c^2).$$

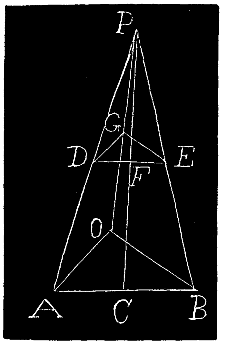
111. Proposed by **G. B. M. ZERR, A. M., Ph. D.,** Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

(a). Find the dimensions of a cup, capacity c, in the form of a frustum of a regular pyramid of n faces, so that its internal surface is a minimum.

(b). Find the dimensions of a cup, capacity c, in the form of a frustum of a hyperbolioid or of a paraboloid, whichever it is, so that its internal surface is a minimum.

Solution by the PROPOSER.

(a). Let $PO=x$, $PG=y$, $\angle AOB=\angle DGE=2\pi/n$, $AO=r$, where O is the center of the circle circumscribing the larger base and G the center of the circle circumscribing the smaller base.



Then $DG=ry/x$, $AB=2r\sin(\pi/n)$, $DE=(2ry/x)\sin(\pi/n)$. Area $DGE=(r^2y^2/x^2)\sin(\pi/n)\cos(\pi/n)$.

$\therefore$ Area of upper base$=(nr^2y^2/x^2\sin(\pi/n)\cos(\pi/n)$.

$PC=\sqrt{(PA^2-AC^2)}=\sqrt{[x^2+r^2-r^2\sin^2(\pi/n)}$

$=\sqrt{[x^2+r^2\cos^2(\pi/n)}$.

Similarly $DF=(y/x)\sqrt{[x^2+r^2\cos^2(\pi/n)]}$.

Area $ADEB=r\sin(\pi/n)\sqrt{[x^2+r^2\cos^2(\pi/n)]}[(x^2-y^2)/x^2]$.

The surface is the least when the upper or smaller base is the bottom of the cup.

Total surface of cup$=u$, volume$=c$.

$$\therefore\ u=nr\sin(\pi/n)\sqrt{[x^2+r^2\cos^2(\pi/n)]}\left(\frac{x^2-y^2}{x^2}\right)+\frac{nr^2y^2}{x^2}\sin(\pi/n)\cos(\pi/n)$$

$c=\frac{1}{3}nr^2\sin(\pi/n)\cos(\pi/n)\left(\frac{x^3-y^3}{x^2}\right)$. Let $(r/x)\cos(\pi/n)=\tan\theta$.

$\therefore u=n\tan\theta\sec\theta\sec(\pi/n)\sin(\pi/n)(x^2-y^2)+ny^2\tan^2\theta\sec(\pi/n)\sin(\pi/n)$. (1).

$c=\frac{1}{3}n\tan^2\theta\sec(\pi/n)\sin(\pi/n)(x^3-y^3)$ .... (2).

Differentiating (1) and (2) we get

$$\frac{dx}{dy}=\left(\frac{\sec\theta-\tan\theta}{\sec\theta}\right)\frac{y}{x}=(1-\sin\theta)\frac{y}{x}\dots.(3). \qquad \frac{dx}{dy}=\frac{y^2}{x^2}\dots.(4).$$

$$\frac{dx}{d\theta}=-\frac{(\sec^2\theta+\tan^2\theta)(x^2-y^2)+2y^2\sec\theta\tan\theta}{2x\tan\theta}\dots.(5).$$

$$\frac{dx}{d\theta}=-\frac{2\sec^2\theta(x^3-y^3)}{3x^2\tan\theta}\dots.(6).$$

From (3) and (4) $y=x(1-\sin\theta)\dots.(7)$.

From (5) and (6), $\sec^2\theta+\tan^2\theta)(x^2-y^2)+2y^2\sec\theta\tan\theta=\frac{4\sec^2\theta(x^3-y^3)}{3x}$

This reduces to $(1+\sin^2\theta)(x^2-y^2)+2y^2\sin\theta=4(x^3-y^3)/3x\dots.(8)$.

(7) in (8) gives $(3-8\sin\theta+3\sin^2\theta)\sin\theta=0$.
$\therefore\ \sin\theta=0$ or $\frac{1}{3}(4\pm\sqrt{7})$.
$\sin\theta$ cannot be zero nor greater than unity.
$\therefore\ \sin\theta=\frac{1}{3}(4-\sqrt{7})\dots.(9)$.
$\therefore\ \tan\theta=\sqrt{\frac{4\sqrt{7}-7}{14}}\dots(10)$.
From (7) and (9), $3y=x(\sqrt{7}-1)\dots.(11)$.

(10) and (11) in (2) gives $x=\frac{1}{3}\left(\frac{2(38\sqrt{7}+89)c}{n\tan(\pi/n)}\right)^{\frac{1}{2}}$.

From (11), $y=\frac{1}{3}x(\sqrt{7}-1)=\frac{1}{3}\left(\frac{4(\sqrt{7}+13)c}{n\tan(\pi/n)}\right)^{\frac{1}{2}}$.

$x-y=x-\frac{1}{3}x(\sqrt{7}-1)=\frac{1}{3}x(4-\sqrt{7})=\left(\frac{2(\sqrt{7}-2)c}{n\tan(\pi/n)}\right)^{\frac{1}{2}}=$altitude.

$r=x\tan\theta\sec(\pi/n)=\left(\frac{(91+88\sqrt{7})c^2}{98n^2\tan^2(\pi/n)}\right)^{\frac{1}{4}}\sec(\pi/n)$.

$AB=2r\sin(\pi/n)=2\left(\frac{(91+88\sqrt{7})c^2}{98n^2\tan^2(\pi/n)}\right)^{\frac{1}{4}}\tan(\pi/n)$, side lower base.

$DE=\frac{2ry}{x}\sin(\pi/n)=2\left(\frac{4(11\sqrt{7}-28)c^2}{49n^2\tan^2(\pi/n)}\right)^{\frac{1}{4}}\tan(\pi/n)$, side upper base.

(2). Let $y^2=4ax$ be the equation to the parabola, then we get from the Integral Calculus the two equations, between the limits x_2 and x_1,

$$u=\tfrac{8}{3}\pi\sqrt{a}[(x_2+a)^{\frac{3}{2}}-(x_1+a)^{\frac{3}{2}}]+4\pi a x_1\dots.(1).$$
$$c=2\pi a(x_2^2-x_1^2)\dots.(2).$$

From (1) and (2), by differentiation, we get

$$\frac{dx_2}{dx_1}=\frac{(x_1+a)^{\frac{1}{2}}-\sqrt{a}}{(x_2+a)^{\frac{1}{2}}}\ldots.(3);\quad \frac{dx_2}{dx_1}=\frac{x_1}{x_2}\ldots.(4).$$

$$\frac{dx_2}{da}=-\frac{(x_2+a)^{\frac{3}{2}}-(x_1+a)^{\frac{3}{2}}+3a(x_2+a)^{\frac{1}{2}}-3a(x_1+a)^{\frac{1}{2}}+3\sqrt{a}\,x_1}{3a(x_2+a)^{\frac{1}{2}}}\ldots.(5).$$

$$\frac{dx_2}{da}=-\frac{x_2^2-x_1^2}{2ax_2}\ldots.(6).$$

From (3) and (4) by eliminating dx_2/dx_1 we get

$$x_1=\frac{x_2^2-2x_2\sqrt{[a(x_2+a)]}}{x_2+a}\text{ and } x_1=0\ldots.(7).$$

Eliminating dx_2/da between (5) and (6) and substituting the first value of x_1 from (7) in the resulting equation, we get after reduction

$$36x_2^3-100ax_2^2-279a^2x_2-144a^3=0\ldots\ (8).$$

Let $a/x_2=z$ and (8) becomes

$$144z^3+279z^2+100z-36=0\ \ldots(9).$$

Let $z=v-\frac{31}{48}$ and (9) becomes

$$v^3-\tfrac{12833}{2304}v-\tfrac{88333}{55296}=0\ldots.(10).$$

$\therefore\ v_1=.861506,\ z_1=.215673.$
$v_2=-.445379,\ z_2=1.094212.$
$v_3=-.416138,\ z_3=-1.061971.$
$\therefore\ a=.215673x_2,\ -1.094212x_2$, or $-1.061971x_2$.

a cannot be negative. The first value of a gives $x_1=-.019813x_2$, a negative value and therefore not admissible. From this we learn that the second value of x_1 in (7), $x_1=0$ is the possible value and the cup is not a frustum of a paraboloid, but a paraboloid, a cup with a curved bottom.

$x_1=0$ in (5) and (6) $x_2^2-15ax_2+48a^2=0$.

$\therefore\ x_2=\dfrac{15\pm\sqrt{33}}{2}a.\quad \therefore\ x_2=\dfrac{15-\sqrt{33}}{2}a\ldots(11)$ is the admissible value.

From (11) and (2), $x_2=\left(\dfrac{(15-\sqrt{33})c}{4\pi}\right)^{\frac{1}{3}}$=altitude of cup.

$$a=\left(\frac{(43+5\sqrt{33})c}{3072\pi}\right)^{\frac{1}{3}}=\left(\frac{2c}{\pi(15-\sqrt{33})^2}\right)^{\frac{1}{3}}.$$

$y_2=2\sqrt{ax_2}=2\left(\dfrac{(15+\sqrt{33})c^2}{384\pi^2}\right)^{\frac{1}{6}}$ radius of top.

$$u=\tfrac{1}{96}\left(\frac{(1337+215\sqrt{33})4\pi c^2}{9}\right)^{\frac{1}{3}}[(3298-450\sqrt{33})^{\frac{1}{3}}-2].$$

MECHANICS.

116. Proposed by C. L. CHILTON, Greensboro, Ala.

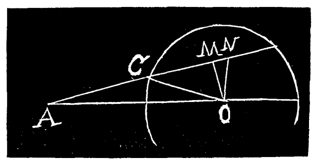
Given, the shaft ABC attached at one end by a pivot to the piston-rod of an engine (at A) and the other to a crank of the wheel CDE (at C). The shaft moves through the distance of two feet in one second from A to B and at the same time turns the crank from C to E. The force propelling the shaft along the constrained course from A to B is 5760 pounds. The mass of the rod and wheel and friction being not considered, what would be the kinetic energy of the machine? or the sum of the moment around O, the center of the wheel?

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let AC, the connecting rod$=l$, $OC=r=$one foot, $\angle AOC=\theta$, force 5760 pounds along $AO=P$, component of P along $AC=Q$. Then moment of crank effect about $O=Q.OM$. In the right triangles AOM, AON, $AO:OM=AN:ON$.

$\therefore P:OM=Q:ON$.

$\therefore Q.OM=P.ON$.



Also $ON:OC=\sin OCN:\sin ONC$.

Let $\angle OAC=\varphi$. Then $\angle ONC=\frac{1}{2}\pi-\varphi$, $\angle OCN=\theta+\varphi$.

$$\therefore ON=\frac{r\sin(\theta+\varphi)}{\cos\varphi}=r\sin\theta+\frac{r\cos\theta\sin\varphi}{\cos\varphi}.$$

$$\text{but } \sin\varphi=\frac{r\sin\theta}{l}. \quad \therefore ON=r\sin\theta+\frac{r^2\sin\theta\cos\theta}{\sqrt{(l^2-r^2\sin^2\theta)}}.$$

$$\therefore \text{moment}=Pr\sin\theta\left(1+\frac{r\cos\theta}{\sqrt{(l^2-r^2\sin^2\theta)}}\right).$$

Now θ varies from 0 to π.

$$\therefore \text{Average moment}=\frac{Pr\int_0^\pi \sin\theta\left(1+\frac{r\cos\theta}{\sqrt{(l^2-r^2\sin^2\theta)}}\right)d\theta}{\int_0^\pi d\theta}=\frac{2Pr}{\pi},$$

a result independent of the connecting rod.

$2Pr/\pi=.6366Pr=3666.816$ or 3667 pounds.

117. Proposed by F. P. MATZ. M. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pa.

How much lower must *one end* of a heavy uniform chain, wound round the circumference of a perfectly rough vertical wheel, hang than *the other end*, when the chain is on the point of motion?